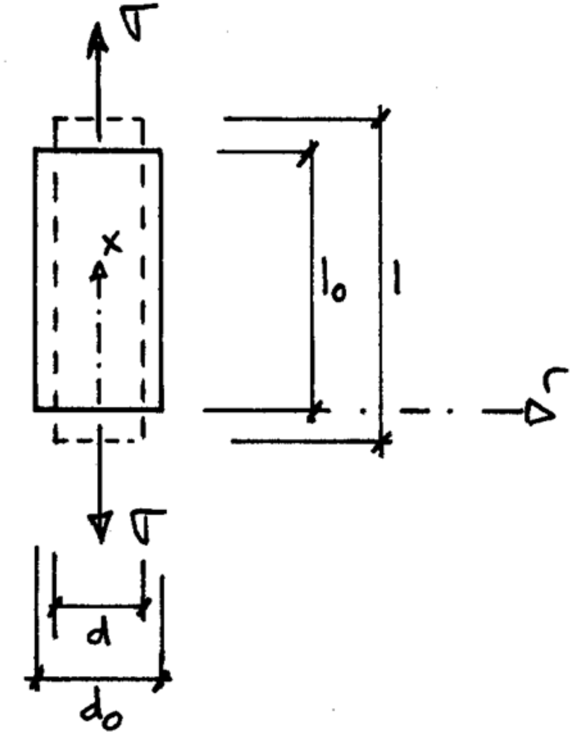

Geomechanics

LECTURE 3

ELASTICITY

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Laboratory of soil mechanics – Fall 2025

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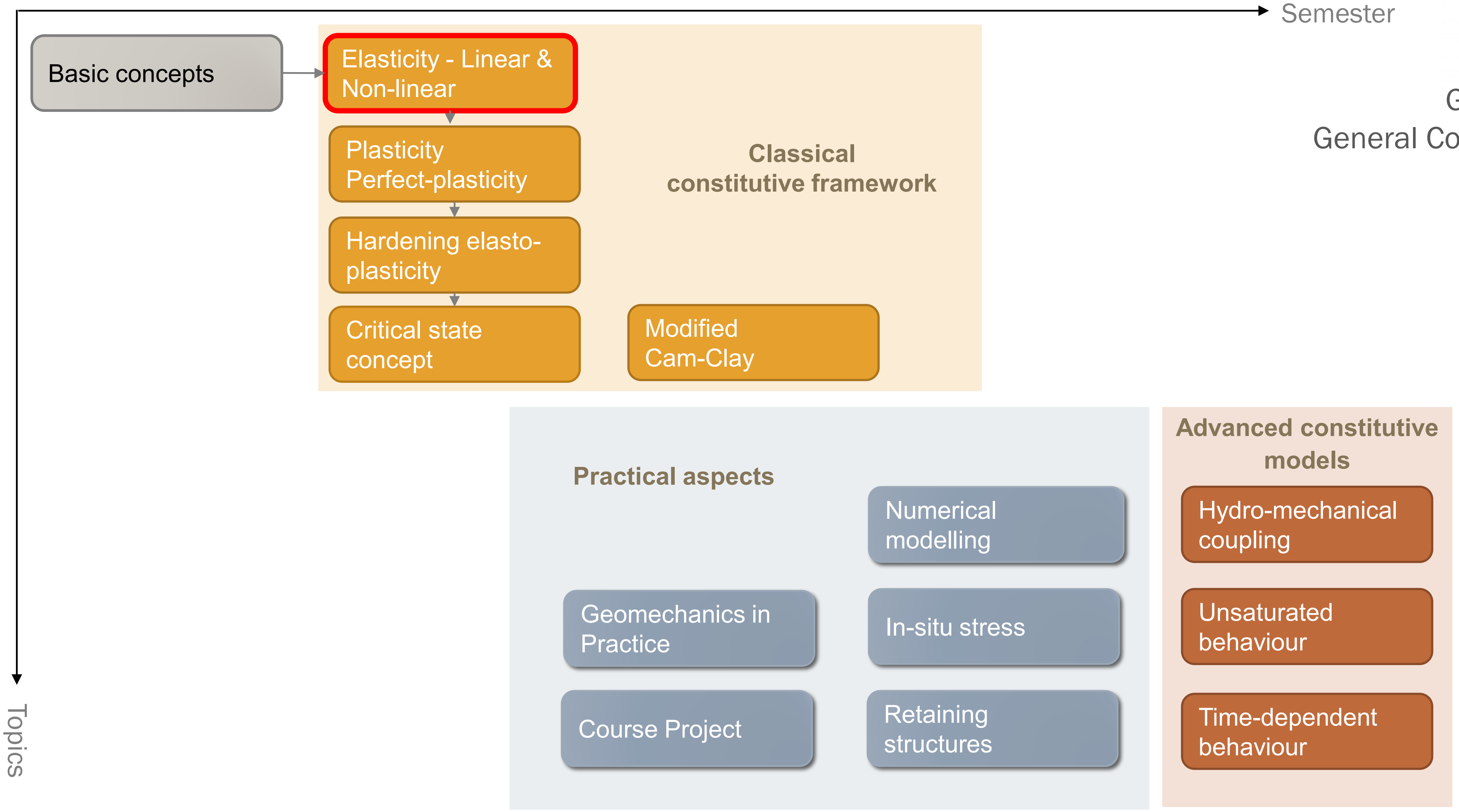
Geomechanics
General Course overview



Content

- Constitutive modelling
- Linear elasticity
- Non-linear elasticity

Constitutive modelling

GENERAL CONSIDERATIONS

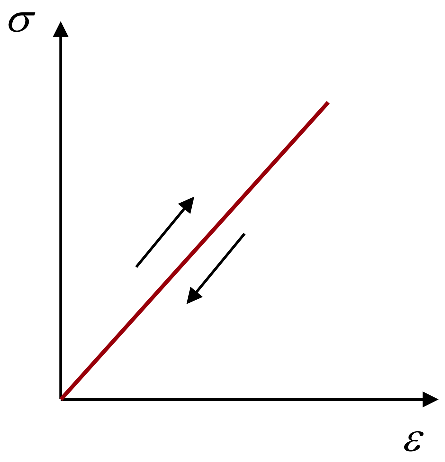
CONSTITUTIVE EQUATIONS

General considerations

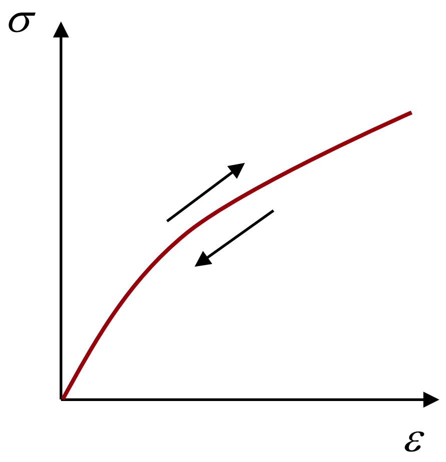
- The constitutive response expresses the link between changes in (effective) stresses and changes in strains.
- Many models (hundreds!) exist which differ for the the mathematical framework, the choice of the state variables, the physical phenomena which are reproduced.
- Some models are widely used so that they are generally available in all numerical codes intended for geotechnical applications: **isotropic elasticity**, **elastic-perfectly plastic Mohr-Coulomb**, and **Cam clay**.
- The user has the responsibility to select the model to be used for the analysis.
- Awareness is needed of the particular features of soil history and soil response that are likely to be important in a particular application and ensure that the constitutive model that is adopted is indeed able to reproduce these features.
- In all modelling, **adequate complexity** should be sought.

General considerations

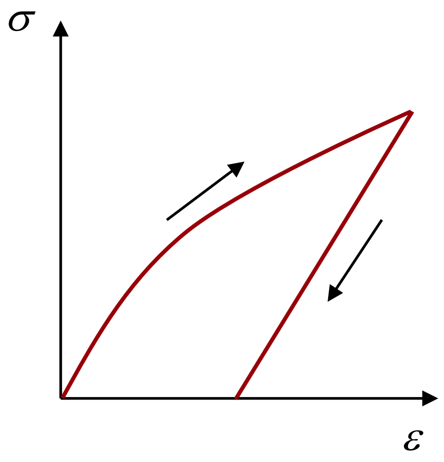
- The fully reversible response upon unloading is the key element of elasticity. It is not possible to assess if the behaviour is elastic if we do not check for permanent deformations once the load is removed.
- The various types of constitutive models can be identified according to the behaviour upon unloading.



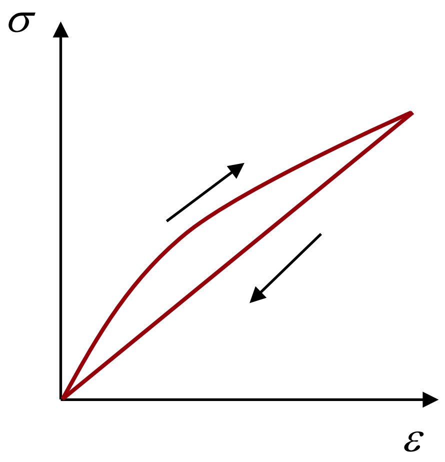
Linear elasticity



Nonlinear elasticity



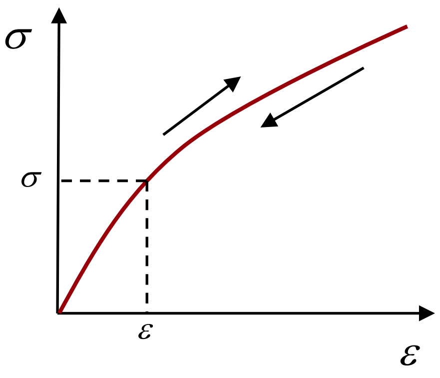
Elasto-plasticity



Damage

General considerations

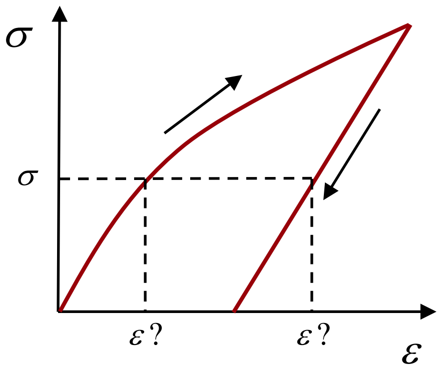
- Constitutive behavior depends on effective stresses.
- A one-to-one relationship between effective stresses and strains exists only in limited cases (e.g. elasticity).
- In the general case, the constitutive model must be formulable in an incremental form.



Elastic behaviour

One-to-one relationship

$$\sigma = f(\varepsilon) \text{ or}$$
$$\varepsilon = f(\sigma)$$



Elasto-plastic behaviour

Relationship between σ and ε depends on the stress path

General statement of constitutive equations

- Stress-strain relationship in incremental form:

$d(.)$ Denotes the increment of (.)

- General equation of stress-strain can be expressed:

$$d\sigma'_{ij} = D_{ijkl} d\varepsilon_{kl}$$

Constitutive tensor rank 4:
3^4=81 components

$$d\sigma'_{11} = D_{1111}d\varepsilon_{11} + D_{1112}d\varepsilon_{12} + D_{1113}d\varepsilon_{13} + D_{1121}d\varepsilon_{21} + D_{1122}d\varepsilon_{22} + D_{1123}d\varepsilon_{23} + D_{1131}d\varepsilon_{31} + D_{1132}d\varepsilon_{32} + D_{1133}d\varepsilon_{33}$$

Independent stress-strain components

- Since the stress and strain tensors are symmetric (only 6 independent components), a vectorial representation is often used (Voigt's form):

$$[\sigma_{11} \quad \sigma_{22} \quad \sigma_{33} \quad \sigma_{23} \quad \sigma_{13} \quad \sigma_{12}]$$

$$[\varepsilon_{11} \quad \varepsilon_{22} \quad \varepsilon_{33} \quad \varepsilon_{23} \quad \varepsilon_{13} \quad \varepsilon_{12}]$$

- The general stress-strain relationship becomes

$$\begin{bmatrix} d\sigma'_{11} \\ d\sigma'_{22} \\ d\sigma'_{33} \\ d\sigma'_{23} \\ d\sigma'_{13} \\ d\sigma'_{12} \end{bmatrix} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & D_{1123} & D_{1113} & D_{1112} \\ & D_{2222} & D_{2233} & D_{2223} & D_{2213} & D_{2212} \\ & & D_{3333} & D_{3323} & D_{3313} & D_{3312} \\ & & & D_{2323} & D_{2313} & D_{2312} \\ & sym. & & & D_{1313} & D_{1312} \\ & & & & & D_{1212} \end{bmatrix} \begin{bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\varepsilon_{23} \\ d\varepsilon_{13} \\ d\varepsilon_{12} \end{bmatrix}$$

Number of Independent components: 6 21 6

Linear Elasticity

ELASTIC CONSTANTS

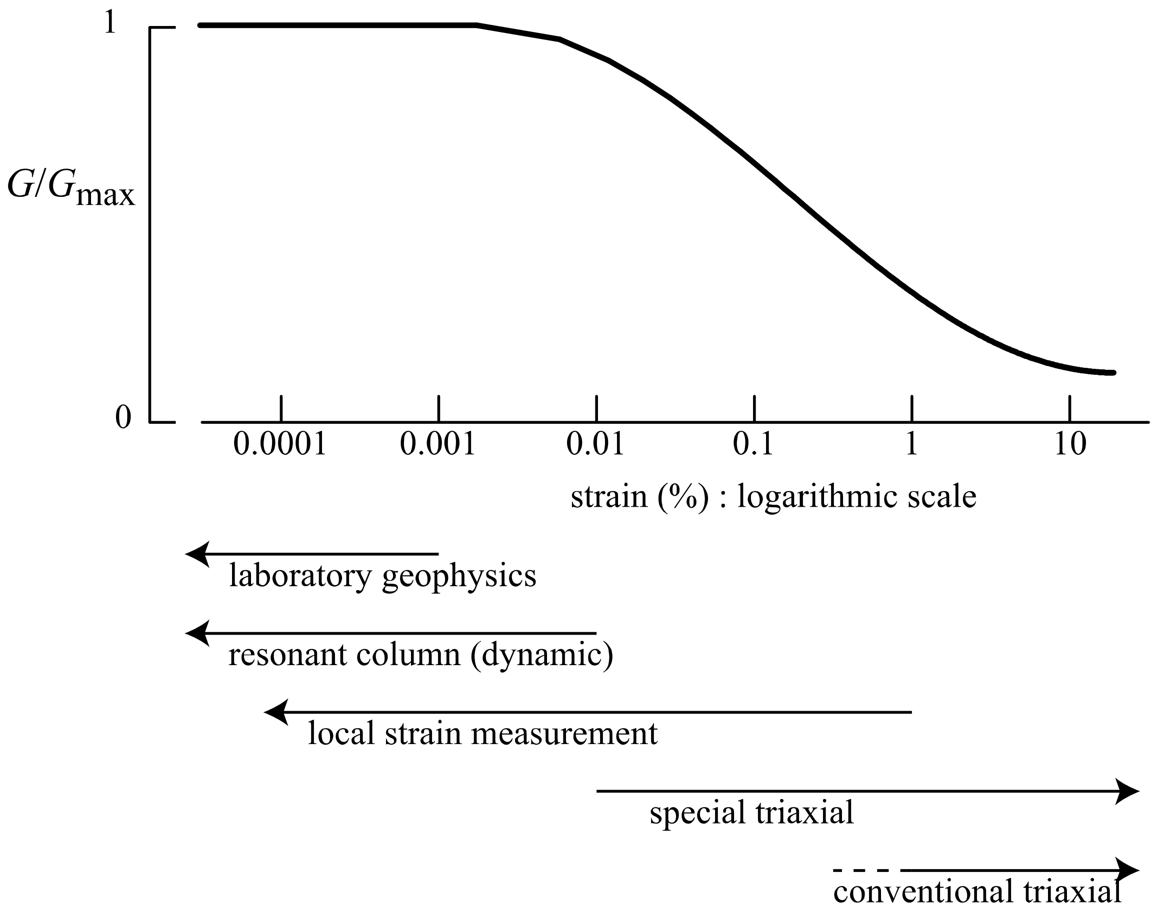
GENERALIZED HOOKE'S EQUATION

TRIAXIAL STRESS CASE

ANISOTROPY

Elasticity - Generalities

- D. M. Wood. Soil behaviour and critical state soil mechanics. Cambridge University Press. 1990.

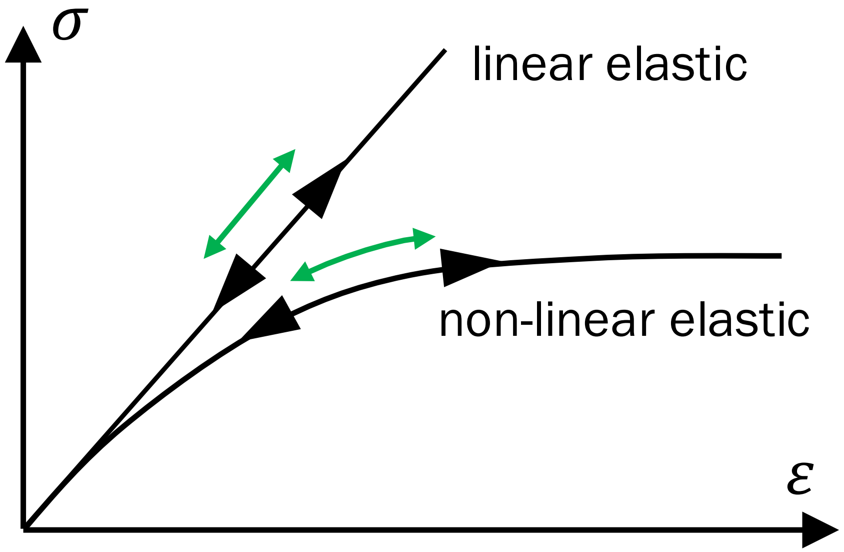


Elasticity - Generalities

- Elasticity – simple definition : Fully reversible deformation

- Linear Elasticity
- Non-linear elasticity

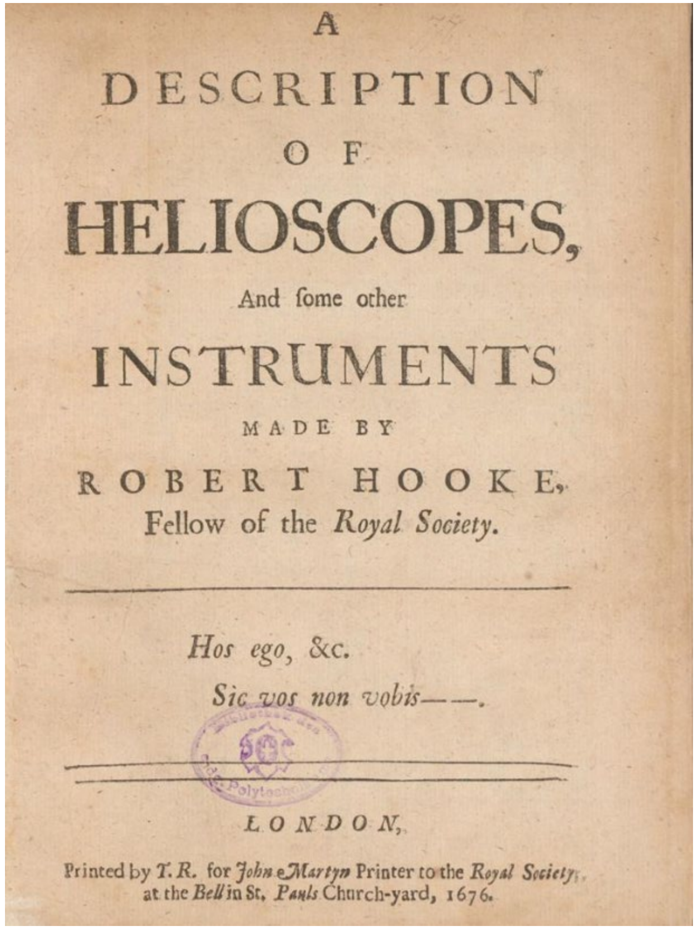
- Isotropic elasticity
- Anisotropic elasticity



Isotropic linear elasticity

Robert Hooke (1635-1703)

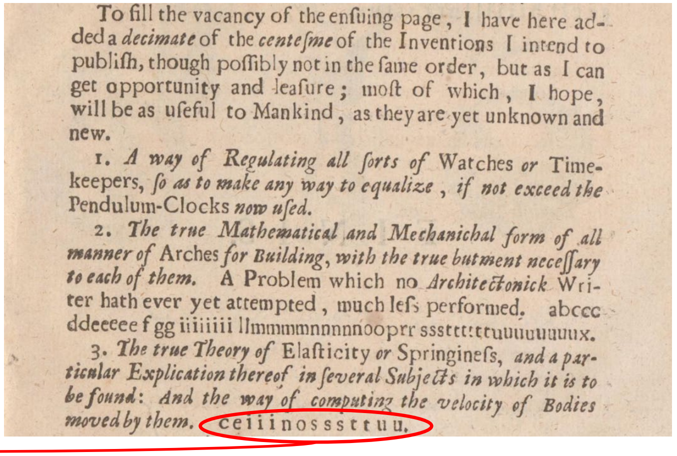
- English scientist active in many fields
- Author of the Hooke’s law (object of this lecture)
- Pioneer in Microscopy: author of *Micrographia* (1665)
- After Great Fire of London in 1666, he collaborated in the reconstruction of the city



A
DESCRIPTION
OF
HELIOSCOPES,
And ſome other
INSTRUMENTS
MADE BY
ROBERT HOOKE,
Fellow of the *Royal Society*.

Hos ego, &c.
Sic vos non vobis——.

LONDON,
Printed by *T. R.* for *John Martyn* Printer to the *Royal Society*, at the *Bell* in St. *Pauls* Church-yard, 1676.



To fill the vacancy of the enſuing page, I have here added a *decimate* of the *centeſme* of the Inventions I intend to publiſh, though poſſibly not in the ſame order, but as I can get opportunity and leaſure; moſt of which, I hope, will be as uſeful to Mankind, as they are yet unknown and new.

1. *A way of Regulating all ſorts of* Watches *or* Time-keepers, *ſo as to make any way to equalize, if not exceed the* Pendulum-Clocks *now uſed.*

2. *The true Mathematical and Mechanichal form of all manner of* Arches *for Building, with the true butment neceſſary to each of them.* A Problem which no *Architectonick* Writer hath ever yet attempted, much leſs performed. abccc ddeeeee f gg iiiiiii llmmmmnnnnnooprr sssttttttuuuuuuuuux.

3. *The true Theory of* Elaſticity *or* Springineſs, *and a particular Explication thereof in ſeveral Subjects in which it is to be found: And the way of computing the velocity of Bodies moved by them.* ceiiinosssttuu.

“Ut tensio, sic vis”

Isotropic linear elasticity

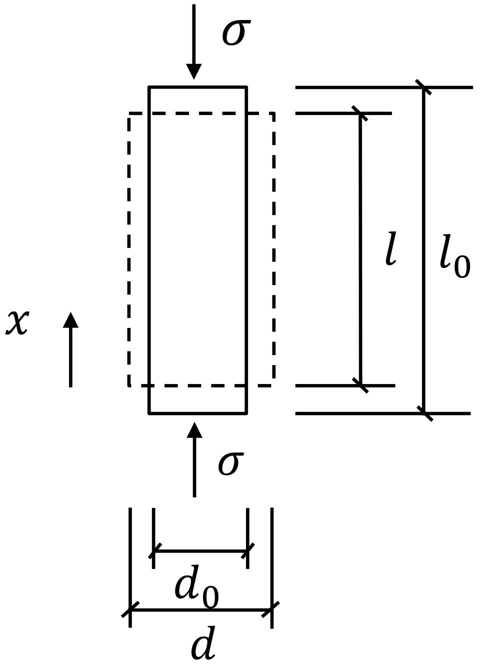
- Hooke's law in 1D

Elastic Young modulus

$$\sigma = E \cdot \varepsilon_x$$

$$\varepsilon_r = -\nu \cdot \varepsilon_x$$

Poisson ratio



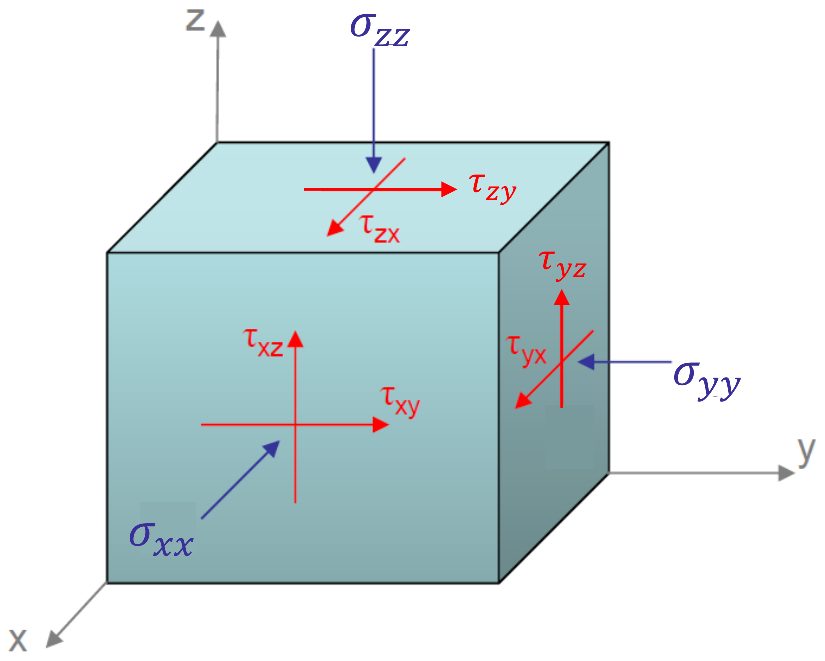
$$\varepsilon_x = -\frac{l - l_0}{l_0}$$

$$\varepsilon_r = -\frac{d - d_0}{d_0}$$

Isotropic linear elasticity

- Elastic relation in 3D

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\left(\sigma_{yy} + \sigma_{zz}\right)\right]$$
$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\left(\sigma_{xx} + \sigma_{zz}\right)\right]$$
$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{xx} + \sigma_{yy}\right)\right]$$
$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$
$$\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz}$$
$$\gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx}$$



Generalized Hooke's equation in 3D

Stiffness form in terms of Young modulus E and Poisson coefficient ν :

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & 1-2\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1-2\nu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

Tensor notation:

$$\sigma'_{ij} = \frac{E}{(1+\nu)} \left[\frac{\nu}{1-2\nu} \varepsilon_{kk} \delta_{ij} + \varepsilon_{ij} \right]$$

Generalized Hooke's equation in 3D

Compliance form in terms of Young modulus E and Poisson coefficient ν :

$$\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{xy}\\ \varepsilon_{yz}\\ \varepsilon_{zx}\end{bmatrix}=\frac{1}{E}\begin{bmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{bmatrix}\begin{bmatrix}\sigma'_{xx}\\ \sigma'_{yy}\\ \sigma'_{zz}\\ \sigma'_{xy}\\ \sigma'_{yz}\\ \sigma'_{zx}\end{bmatrix}$$

Generalized Hooke’s equation in 3D

Stiffness form in terms of bulk modulus K and shear modulus G :

$$K = \frac{E}{3(1-2\nu)} \qquad G = \frac{E}{2\left(1+\nu\right)}$$

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{bmatrix} = \begin{bmatrix} K+\frac{4G}{3} & K-\frac{2G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K+\frac{4G}{3} & K-\frac{2G}{3} & 0 & 0 & 0 \\ K-\frac{2G}{3} & K-\frac{2G}{3} & K+\frac{4G}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2G & 0 & 0 \\ 0 & 0 & 0 & 0 & 2G & 0 \\ 0 & 0 & 0 & 0 & 0 & 2G \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

Tensor notation:

$$\sigma'_{ij} = \left(K - \frac{2G}{3}\right)\varepsilon_{kk}\delta_{ij} + 2G\varepsilon_{ij}$$

Generalized Hooke’s equation in 3D

Compliance form in terms of bulk modulus K and shear modulus G :

$$K = \frac{E}{3(1-2\nu)} \qquad G = \frac{E}{2(1+\nu)}$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{G+3K}{9GK} & \frac{2G-3K}{18GK} & \frac{2G-3K}{18GK} & 0 & 0 & 0 \\ \frac{2G-3K}{18GK} & \frac{G+3K}{9GK} & \frac{2G-3K}{18GK} & 0 & 0 & 0 \\ \frac{2G-3K}{18GK} & \frac{2G-3K}{18GK} & \frac{G+3K}{9GK} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G} \end{bmatrix} \begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{bmatrix}$$

Generalized Hooke’s equation in 3D

Stiffness form in terms of Lame’s constants λ and μ :

$$\lambda = K - \frac{2G}{3} \qquad \mu = G$$

Tensor notation:

$$\sigma'_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

$$\begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{bmatrix} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix}$$

Generalized Hooke’s equation in 3D

Compliance form in terms of Lame’s constants λ and μ :

$$\lambda = K - \frac{2G}{3} \qquad \mu = G$$

$$\begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \end{bmatrix} = \begin{bmatrix} \frac{\lambda+\mu}{2\mu^2+3\lambda\mu} & \frac{-\lambda}{4\mu^2+6\lambda\mu} & \frac{-\lambda}{4\mu^2+6\lambda\mu} & 0 & 0 & 0 \\ \frac{-\lambda}{4\mu^2+6\lambda\mu} & \frac{\lambda+\mu}{2\mu^2+3\lambda\mu} & \frac{-\lambda}{4\mu^2+6\lambda\mu} & 0 & 0 & 0 \\ \frac{-\lambda}{4\mu^2+6\lambda\mu} & \frac{-\lambda}{4\mu^2+6\lambda\mu} & \frac{\lambda+\mu}{2\mu^2+3\lambda\mu} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2\mu} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2\mu} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2\mu} \end{bmatrix} \begin{bmatrix} \sigma'_{xx} \\ \sigma'_{yy} \\ \sigma'_{zz} \\ \sigma'_{xy} \\ \sigma'_{yz} \\ \sigma'_{zx} \end{bmatrix}$$

Pairs of elastic constants

E Young modulus

$$E = \frac{9KG}{3K + G}$$

ν Poisson ratio

$$\nu = \frac{3K - 2G}{6K + 2G}$$

K Bulk modulus

$$p' = K\varepsilon_v$$

$$K = \frac{E}{3(1 - 2\nu)}$$

G Shear modulus

$$\tau_{xy} = G\gamma_{xy}$$

$$G = \frac{E}{2(1 + \nu)}$$

λ Lamé's constant

$$\sigma_{xx} = \lambda\varepsilon_v + 2\mu\varepsilon_{xx}$$

$$\lambda = \frac{E\nu}{(1 + \nu)\cdot(1 - 2\nu)} = K - \frac{2G}{3}$$

μ Lamé's constant

$$\mu = G = \frac{E}{2(1 + \nu)}$$

Special case: Triaxial test

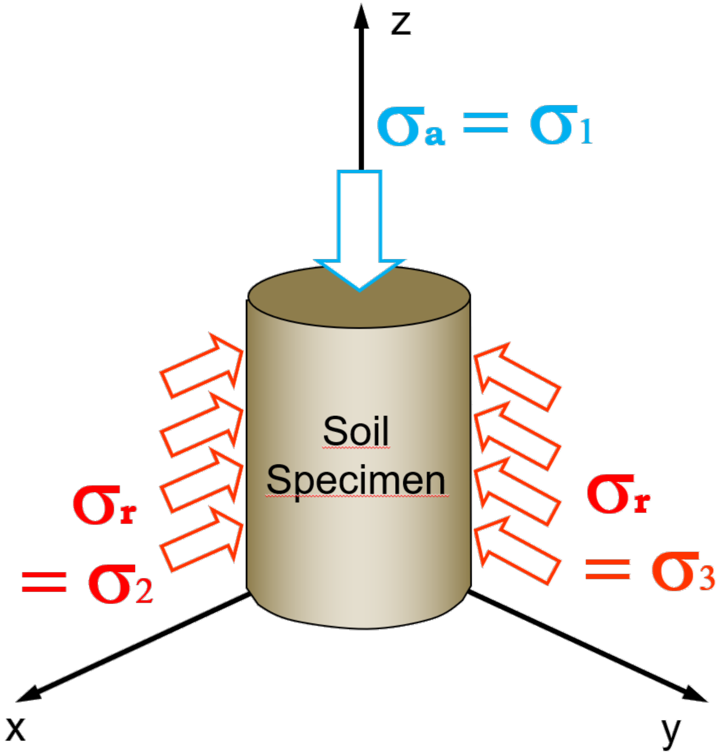
$$\sigma_{ij} = \begin{bmatrix} \sigma_a & 0 & 0 \\ 0 & \sigma_r & 0 \\ 0 & 0 & \sigma_r \end{bmatrix} et\ \varepsilon_{ij} = \begin{bmatrix} \varepsilon_a & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & \varepsilon_r \end{bmatrix}$$

Mean effective stress: $$p' = \frac{\sigma_1' + \sigma_2' + \sigma_3'}{3} = \frac{\sigma'_a + 2\sigma'_r}{3}$$

Volumetric strain: $$\varepsilon_v = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \varepsilon_a + 2\varepsilon_r$$

Deviatoric stress: $$q = q' = \sigma_1 - \sigma_3 = \sigma_a - \sigma_r$$

Deviatoric strain: $$\varepsilon_d = \frac{2}{3}(\varepsilon_1 - \varepsilon_3) = \frac{2}{3}(\varepsilon_a - \varepsilon_r)$$



Isotropic linear elasticity in TX tests

$$d\varepsilon_a = \frac{1}{E}\left[d\sigma_a' - 2\nu d\sigma_r'\right] \qquad d\varepsilon_r = \frac{1}{E}\left[-\nu d\sigma_a' + (1-\nu)d\sigma_r'\right]$$

Compliance form

$$\begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix} = \frac{1}{E}\begin{bmatrix} 1 & -2\nu \\ -\nu & 1-\nu \end{bmatrix}\begin{bmatrix} d\sigma_a' \\ d\sigma'_r \end{bmatrix}$$

Stiffness form

$$\begin{bmatrix} d\sigma_a' \\ d\sigma'_r \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & -2\nu \\ -\nu & 1 \end{bmatrix}\begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix}$$

- Matrices aren't symmetric because the variables are disjointed

- Useful relations:

$$\begin{bmatrix} p' \\ q \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \sigma_a' \\ \sigma_r' \end{bmatrix} \qquad \begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix} = \begin{bmatrix} 1/3 & 1 \\ 1/3 & -1/2 \end{bmatrix}\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix}$$

$$\begin{bmatrix} \sigma_a' \\ \sigma_r' \end{bmatrix} = \begin{bmatrix} 1 & 2/3 \\ 1 & -1/3 \end{bmatrix}\begin{bmatrix} p' \\ q \end{bmatrix} \qquad \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix}$$

Isotropic linear elasticity in TX tests

- In terms of conjugated stress/strain variables:

COMPLIANCE FORM

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix} \rightarrow \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \frac{1}{E} \begin{bmatrix} 1 & -2\nu \\ -\nu & 1-\nu \end{bmatrix} \begin{bmatrix} d\sigma'_a \\ d\sigma'_r \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2/3 & -2/3 \end{bmatrix} \frac{1}{E} \begin{bmatrix} 1 & -2\nu \\ -\nu & 1-\nu \end{bmatrix} \begin{bmatrix} 1 & 2/3 \\ 1 & -1/3 \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 3(1-2\nu) & 0 \\ 0 & \frac{2}{3}(1+\nu) \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} \frac{1}{K} & 0 \\ 0 & \frac{1}{3G} \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

With:

$$K = \frac{E}{3(1-2\nu)}$$ Bulk Modulus

$$G = \frac{E}{2(1+\nu)}$$ Shear Modulus

Isotropic linear elasticity in TX tests

- In terms of conjugated stress/strain variables:

STIFFNESS FORM

$$\begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma'_a \\ \sigma'_r \end{bmatrix} \rightarrow \begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & -1 \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} d\varepsilon_a \\ d\varepsilon_r \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ 1 & -1 \end{bmatrix} \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & 2\nu \\ \nu & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 1 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix}$$

$$\begin{bmatrix} dp' \\ dq \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{(1+\nu)}{3} & 0 \\ 0 & 3\frac{(1-2\nu)}{2} \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix}$$

With:

$$K = \frac{E}{3(1-2\nu)}$$ Bulk Modulus

$$G = \frac{E}{2(1+\nu)}$$ Shear Modulus

Isotropic linear elasticity in TX tests

- Both stiffness and compliance forms show 0 in the non-diagonal terms:
    - Volume change and distortions are uncoupled process
    - Dilatancy?

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1/K & 0 \\ 0 & 1/3G \end{bmatrix} \begin{bmatrix} dp' \\ dq \end{bmatrix}$$

$$\begin{bmatrix} dp' \\ dq \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & 3G \end{bmatrix} \begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix}$$

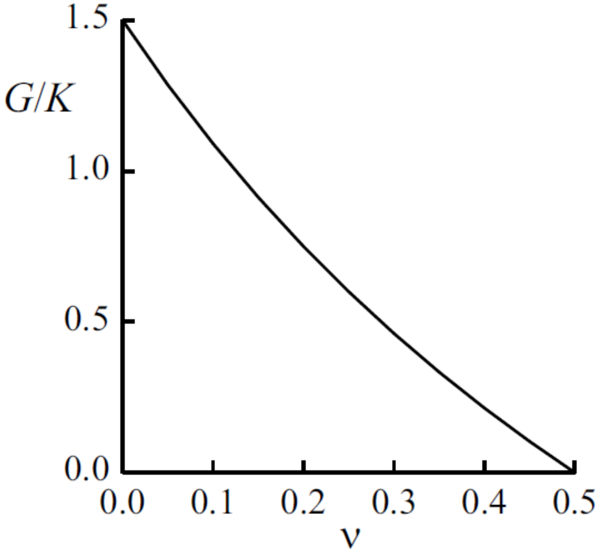
- Only two independent parameters:

$$K = \frac{E}{3(1-2\nu)} \qquad E = \frac{9KG}{G+3K}$$

$$G = \frac{E}{2(1+\nu)} \qquad \nu = \frac{3 - 2\frac{G}{K}}{2\left(\frac{G}{K}+3\right)}$$

$$\Rightarrow \quad \frac{G}{K} = \frac{3(1-2\nu)}{2(1+\nu)}$$

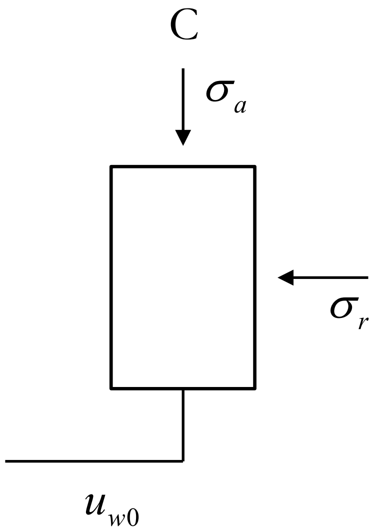
depends only on ν

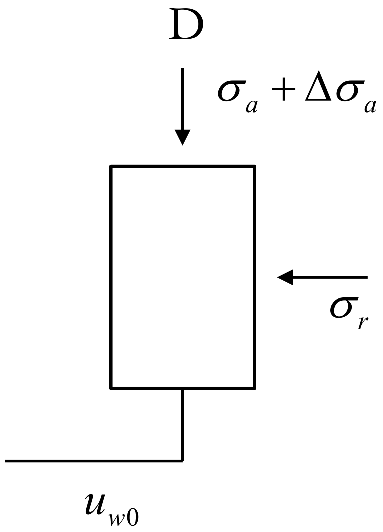


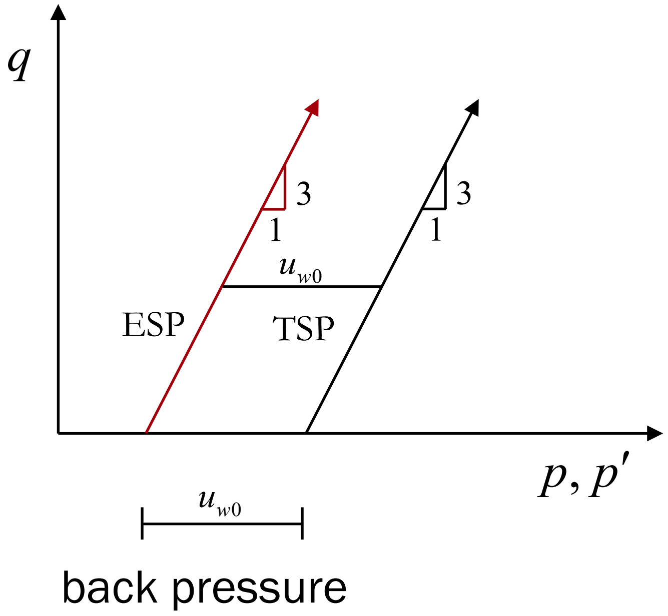
Wood, 2004, p. 102

Isotropic linear elasticity in TX tests

Parameter determination from conventional CD tests







$$p = \frac{\sigma_a + 2\sigma_r}{3}$$

$$p' = \frac{\sigma'_a + 2\sigma'_r}{3}$$

$$q = \sigma_a - \sigma_r = 0$$

$$p \uparrow \quad q \uparrow$$

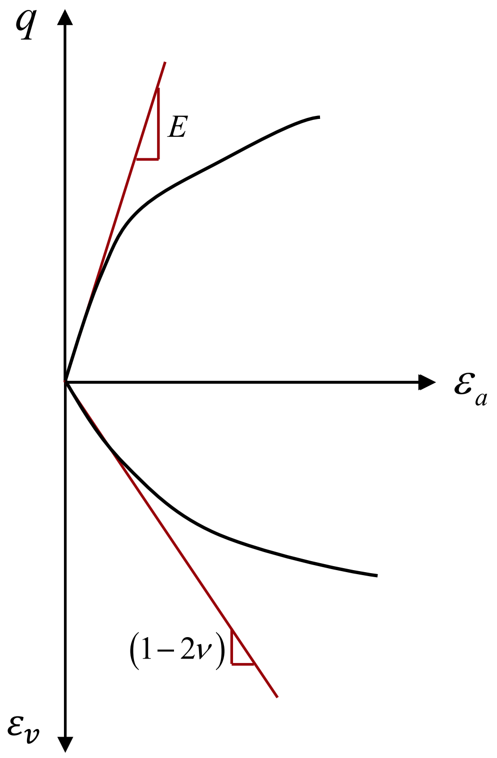
$$\Delta p = \Delta p' = \frac{\Delta\sigma_a}{3}$$

$$\Delta q = \Delta\sigma_a$$

$$\frac{\Delta q}{\Delta p} = \frac{\Delta q}{\Delta p'} = 3$$

Isotropic linear elasticity in TX tests

Elastic parameters from the initial stages of CD triaxial tests



$$as\ d\sigma_r' = 0$$

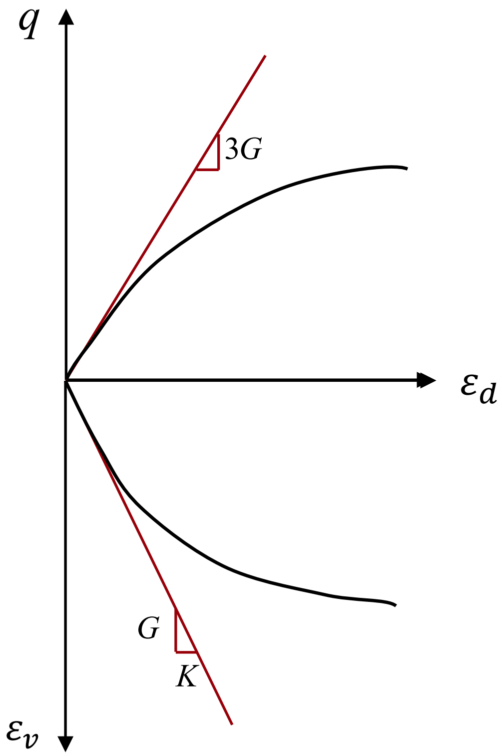
$$dq = d\sigma_a' = d\varepsilon_a E$$

$$\boxed{\rightarrow \frac{dq}{d\varepsilon_a} = E}$$

$$d\varepsilon_v = d\varepsilon_a + 2d\varepsilon_r$$

$$\frac{d\varepsilon_v}{d\varepsilon_a} = 1 + 2\frac{d\varepsilon_r}{d\varepsilon_a}$$

$$\boxed{\rightarrow \frac{d\varepsilon_v}{d\varepsilon_a} = 1 - 2\nu}$$



$$d\varepsilon_d = \frac{dq}{3G}$$

$$\boxed{\rightarrow \frac{dq}{d\varepsilon_d} = 3G}$$

$$d\varepsilon_v = \frac{dp'}{K}; d\varepsilon_d = \frac{dq}{3G}$$

$$\frac{d\varepsilon_v}{d\varepsilon_d} = \frac{3G}{K}\frac{dp'}{dq}$$

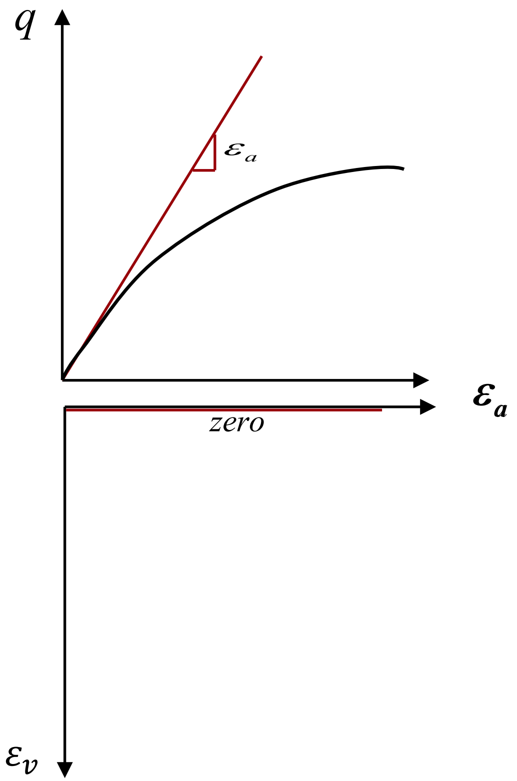
$$\boxed{\rightarrow \frac{d\varepsilon_v}{d\varepsilon_d} = \frac{G}{K}}$$

Isotropic linear elasticity in TX tests

Elastic parameters from CU triaxial tests

No volume change $d\varepsilon_v = 0$ ⇒

$$d\varepsilon_d = \frac{2}{3}\left[d\varepsilon_a - \left(-\frac{d\varepsilon_a}{2}\right)\right] = \frac{2}{3}\frac{3d\varepsilon_a}{2} = d\varepsilon_a \Rightarrow d\varepsilon_d = d\varepsilon_a$$



$$d\varepsilon_d = d\varepsilon_a = \frac{dq}{3G} \rightarrow \boxed{\frac{dq}{d\varepsilon_a} = 3G}$$

$$d\varepsilon_v = 0 = \frac{dp'}{K}$$

$dp' = 0$

or

$K = +\infty$

no reason why $K = +\infty$
(it is a property of the soil skeleton)
so it is $dp' = 0$

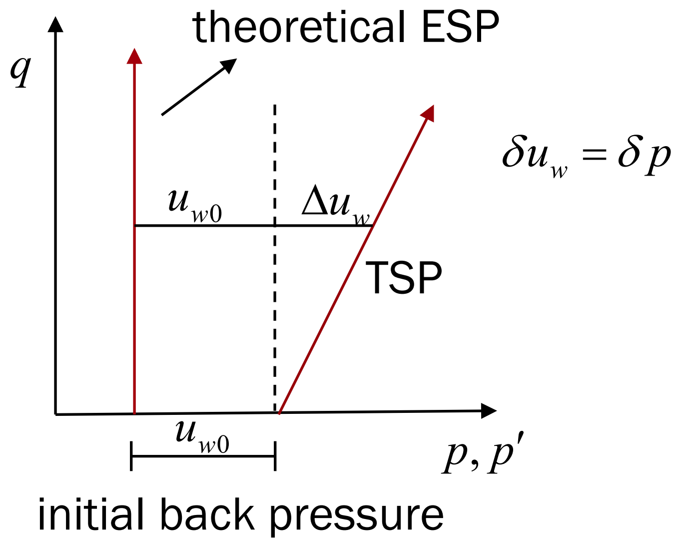
⇓

We cannot determine K from CU

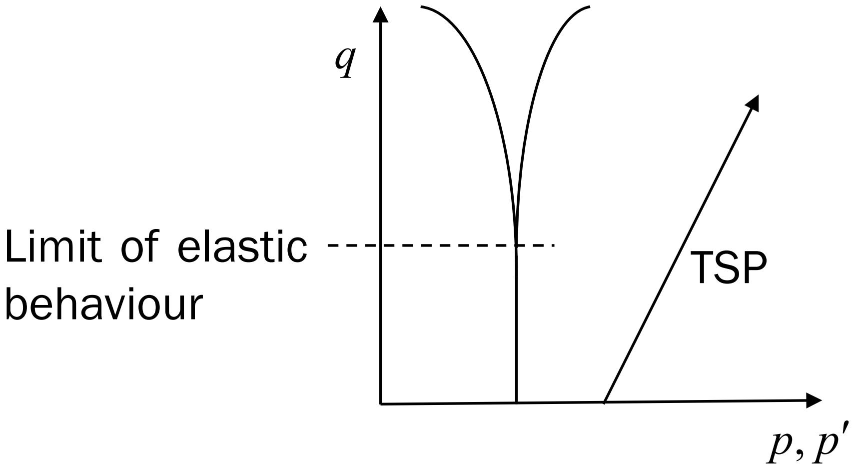
Isotropic linear elasticity in TX tests

Elastic parameters from CU triaxial tests

- The direct link between the increase in mean total stress (dp) and the generated excess pore water pressure (du_w) makes not possible to reproduce a dilatant behavior in undrained shearing.



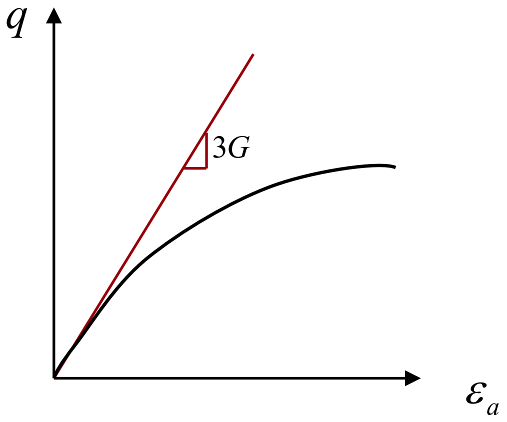
- The $dp' = 0$ concept can be used to identify the limit of the elastic behaviour.

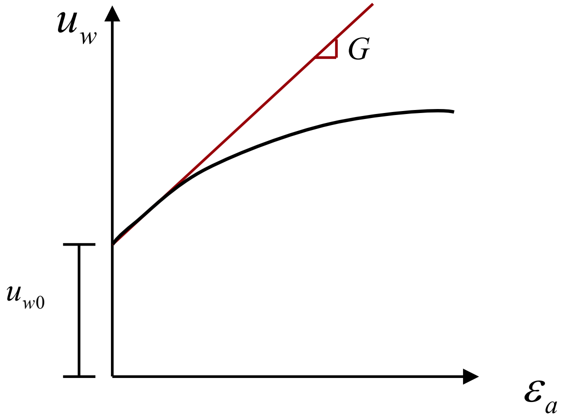


Isotropic linear elasticity in TX tests

Elastic parameters from CU triaxial tests

$$\begin{bmatrix} d\varepsilon_v \\ d\varepsilon_d \end{bmatrix} = \begin{bmatrix} 1/K_u & 0 \\ 0 & 1/3G_u \end{bmatrix} \begin{bmatrix} dp \\ dq \end{bmatrix}$$





1) Undrained bulk modulus $K_u = +\infty \qquad \delta\varepsilon_v = 0$

It means $K_u = +\infty \Rightarrow \dfrac{E_u}{3(1-2\nu_u)} = +\infty \Leftrightarrow \nu_u = 0{,}5$

2) $d\varepsilon_d = \dfrac{1}{3G_u} dq \Leftrightarrow G_u = G$ (same stiffness)

$$G_u = \frac{E_u}{2(1+\nu_u)} = \frac{E_u}{3} = G = \frac{E}{2(1+\nu)}$$

$$E_u = 3G_u = 3G = \frac{3E}{2(1+\nu)}$$

Undrained and drained parameters are linked!

3) Theoretical development of excess water pressure

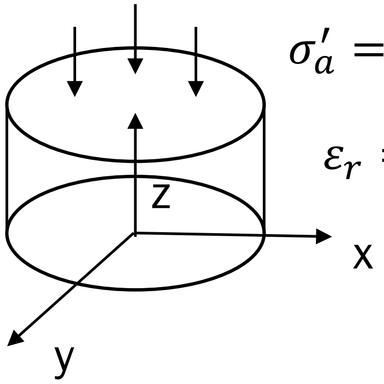
$dp' = 0$ (no volume change)

$dp = du_w = \dfrac{dq}{3}$ (conventional triaxial σ_c = cost)

$$du_w = \frac{1}{3}(3d\varepsilon_d G) = d\varepsilon_a G \rightarrow \frac{du_w}{d\varepsilon_a} = G$$

Isotropic linear elasticity in oedometric tests

- Elastic parameters from oedometric tests



$$\sigma'_a = \sigma'_{zz}$$

$$\varepsilon_r = 0 = \varepsilon_{xx} = \varepsilon_{yy}$$

Mixed control $\begin{cases} \sigma'_{zz} = imposed \\ \varepsilon_{xx} = \varepsilon_{yy} = 0 \end{cases}$

Definition of Oedometric Modulus

$$E_{oed} = \left.\frac{d\sigma'_{zz}}{d\varepsilon_{zz}}\right|_{d\varepsilon_{xx}=d\varepsilon_{yy}=0}$$

$$d\varepsilon_{xx} = 0 = \frac{1}{E}\left[d\sigma'_{xx} - \nu(d\sigma'_{xx} + d\sigma'_{zz})\right]$$

$$d\sigma'_{xx}(1-\nu) - \nu d\sigma'_{zz} = 0$$

$$d\sigma'_{xx} = \frac{\nu}{(1-\nu)} d\sigma'_{zz}$$

Expression for k_0 for an isotropic linear elastic material

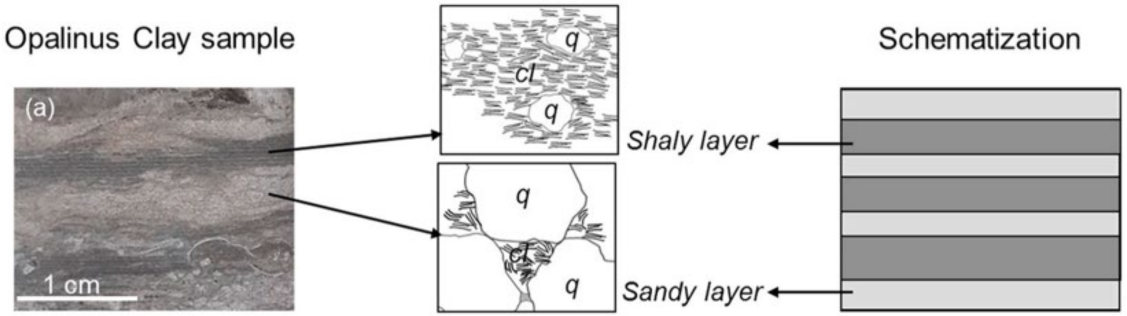
$$k_0 = \frac{\nu}{1-\nu}$$

$$E_{oed} = \frac{d\sigma'_{zz}}{d\varepsilon_{zz}} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} = K + \frac{4}{3}G$$

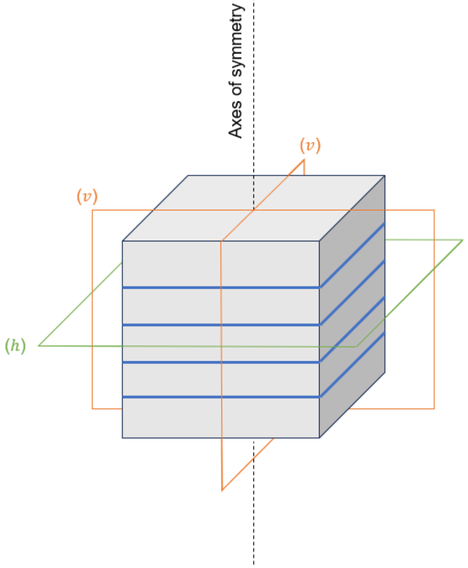
⇒ We need to assume one property (ν for example)

Anisotropic linear elasticity

- Soils deposits are formed in nature under gravity, resulting in possible different soil properties in vertical and horizontal planes.
- A particular type of anisotropy which follows the axial symmetry with respect to any vertical axis is called **cross-anisotropy**.



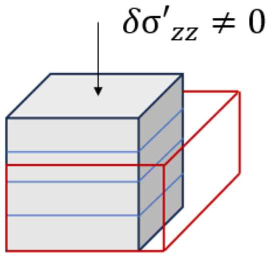
E. Crisci et al. 2021



- The mechanical behaviour in all horizontal planes (h) is identical.
- The mechanical behaviour in all vertical planes (v), passing through the axis of symmetry, is also identical, but different to the one in the horizontal planes.

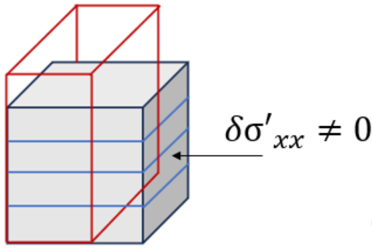
Anisotropic linear elasticity

- For cross-anisotropy, the model has 7 constants: E_v, E_h, ν_{hh}, ν_{vh}, ν_{hv}, G_{hh} and G_{vh}



$E_v = \delta\sigma'_{zz}/\delta\varepsilon_{zz}$

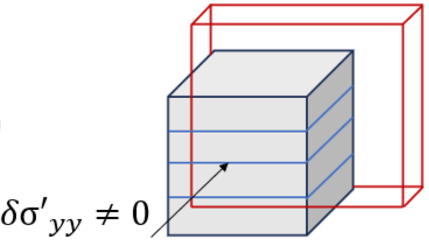
$\nu_{vh} = -\delta\varepsilon_{xx}/\delta\varepsilon_{zz}$
$= -\delta\varepsilon_{yy}/\delta\varepsilon_{zz}$



$E_h = \delta\sigma'_{xx}/\delta\varepsilon_{xx}$

$\nu_{hh} = -\delta\varepsilon_{yy}/\delta\varepsilon_{xx}$

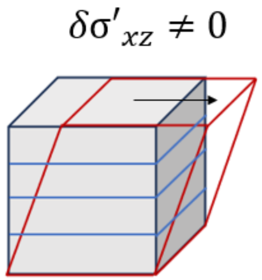
$\nu_{hv} = -\delta\varepsilon_{zz}/\delta\varepsilon_{xx}$



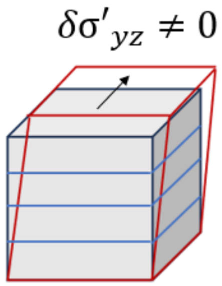
$E_h = \delta\sigma'_{yy}/\delta\varepsilon_{yy}$

$\nu_{hh} = -\delta\varepsilon_{xx}/\delta\varepsilon_{yy}$

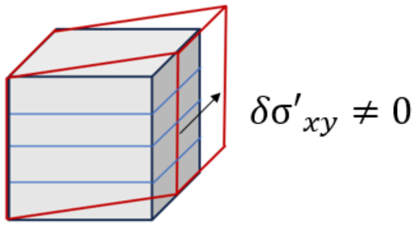
$\nu_{hv} = -\delta\varepsilon_{zz}/\delta\varepsilon_{yy}$



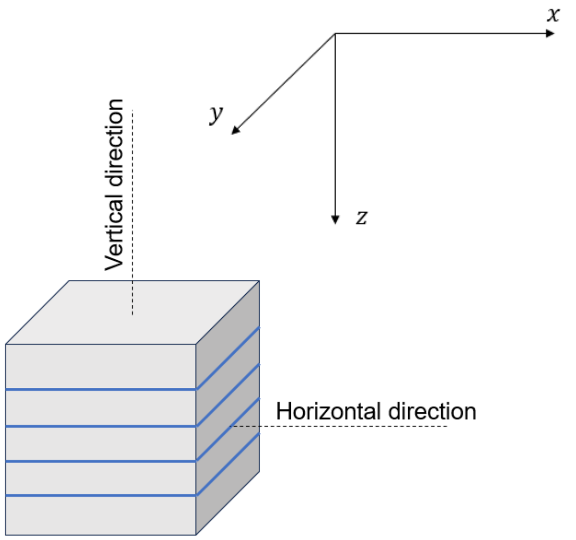
$G_{vh} = \delta\sigma'_{xz}/\delta\gamma_{xz}$



$G_{vh} = \delta\sigma'_{yz}/\delta\gamma_{yz}$



$G_{hh} = \delta\sigma'_{xy}/\delta\gamma_{xy}$



Anisotropic linear elasticity

- Hooke's equations for anisotropy often due to geological formation:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1/E_h & -\nu_{hh}/E_h & -\nu_{vh}/E_v & 0 & 0 & 0 \\ -\nu_{hh}/E_h & 1/E_h & -\nu_{vh}/E_v & 0 & 0 & 0 \\ -\nu_{hv}/E_h & -\nu_{hv}/E_h & 1/E_v & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{vh} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{vh} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{hh} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{xy} \end{Bmatrix}$$

- Sets of parameters: E_v, E_h, ν_{hh}, ν_{vh}, ν_{hv}, G_{hh} and G_{vh}

Anisotropic linear elasticity

- The behaviour in a horizontal plane is isotropic:

$$G_{hh} = \frac{E_h}{2(1+\nu_{hh})}$$

- For thermodynamic reasons, the constitutive matrix must be symmetric:

$$\frac{\nu_{hv}}{E_h} = \frac{\nu_{vh}}{E_v}$$

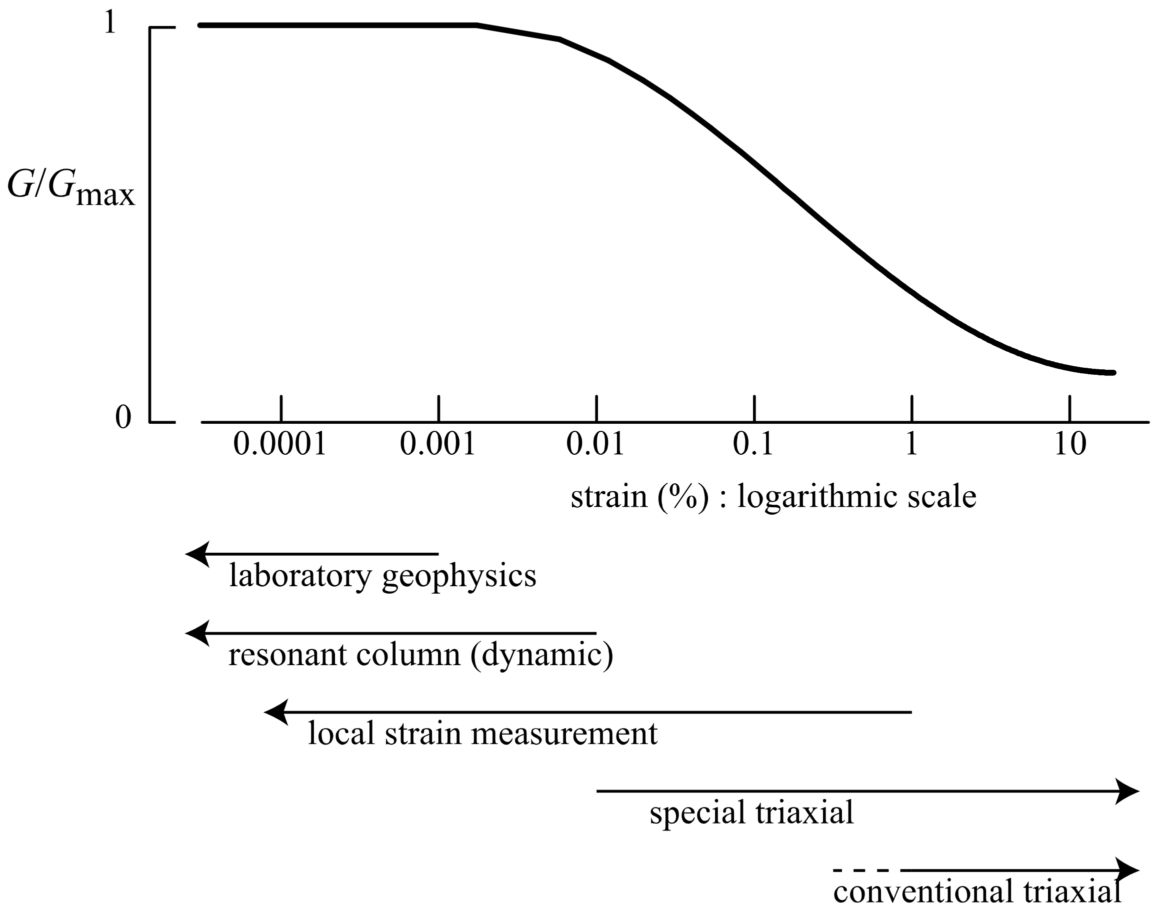
Only <u>5 parameters</u> are independent: E_v, E_h, ν_{hh}, ν_{vh}, and G_{vh}

- If we rewrite the constitutive law for triaxial test and isotropic compression, we obtain 3 independent equations; so <u>additional tests or assumptions</u> are needed to determine the 5 parameters.

Non-Linear Elasticity

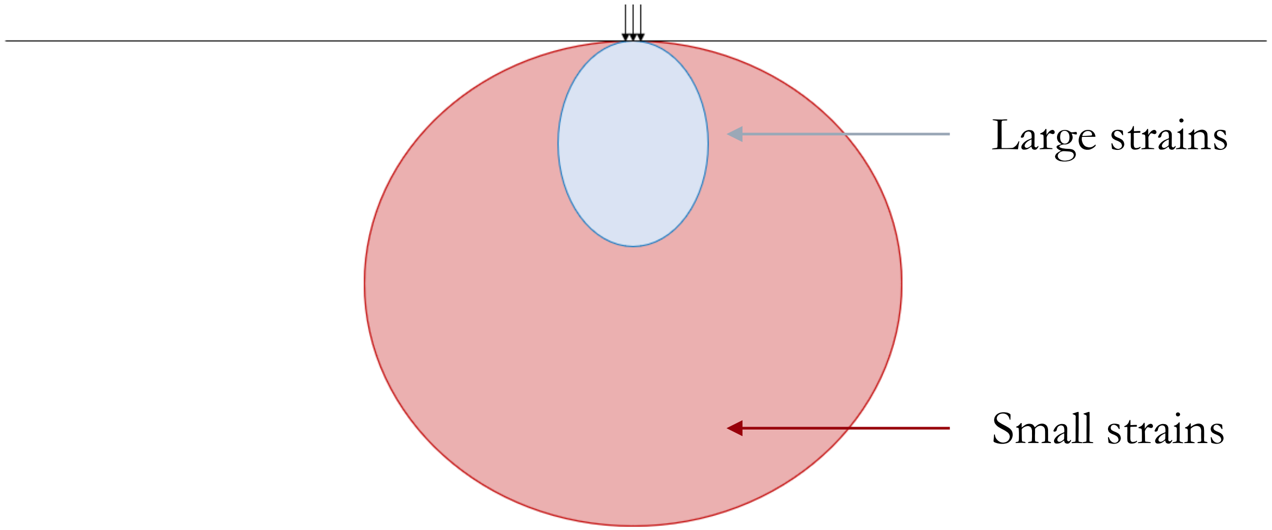
Non-linear elasticity - Generalities

- D. M. Wood. Soil behaviour and critical state soil mechanics. Cambridge University Press. 1990.



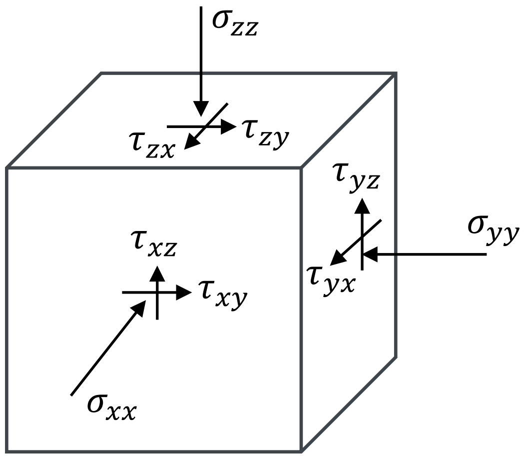
Small strain non-linear elasticity

- In many practical geotechnical problems only a relatively small volume of soil experiences large deformations.
- Assuming the same stiffness independently from the range of strains, would result in computing great displacements.



Three-dimensional wave propagation

Assumptions: homogeneous, elastic, isotropic medium



Equations of motion

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \left(\frac{\partial \sigma_{xx}}{\partial x}\right) + \left(\frac{\partial \tau_{xy}}{\partial y}\right) + \left(\frac{\partial \tau_{xz}}{\partial z}\right) \cancel{+q_x}$$
$$\rho\frac{\partial^2 u_y}{\partial t^2} = \left(\frac{\partial \tau_{xy}}{\partial x}\right) + \left(\frac{\partial \sigma_{yy}}{\partial y}\right) + \left(\frac{\partial \tau_{yz}}{\partial z}\right) \cancel{+q_y}$$
$$\rho\frac{\partial^2 u_z}{\partial t^2} = \left(\frac{\partial \tau_{zx}}{\partial x}\right) + \left(\frac{\partial \tau_{zy}}{\partial y}\right) + \left(\frac{\partial \sigma_{zz}}{\partial z}\right) \cancel{+q_z}$$

Stress-strain relationships

$$\sigma_{xx} = \lambda\varepsilon_v + 2G\varepsilon_{xx} \qquad \tau_{xy} = G\gamma_{xy}$$
$$\sigma_{yy} = \lambda\varepsilon_v + 2G\varepsilon_{yy} \qquad \tau_{xz} = G\gamma_{xz}$$
$$\sigma_{zz} = \lambda\varepsilon_v + 2G\varepsilon_{zz} \qquad \tau_{yz} = G\gamma_{yz}$$

Strain-displacement relationships

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x} \qquad \gamma_{xy} = \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}$$
$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}$$
$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}$$

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$$

Three-dimensional wave propagation

Equations of motion

$$\rho\frac{\partial^2 u_x}{\partial t^2} = (\lambda+G)\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_y}{\partial x\partial y}+\frac{\partial^2 u_z}{\partial x\partial z}\right)+G\left(\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\rho\frac{\partial^2 u_y}{\partial t^2} = (\lambda+G)\left(\frac{\partial^2 u_x}{\partial x\partial y}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_z}{\partial y\partial z}\right)+G\left(\frac{\partial^2 u_y}{\partial x^2}+\frac{\partial^2 u_y}{\partial y^2}+\frac{\partial^2 u_y}{\partial z^2}\right)$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = (\lambda+G)\left(\frac{\partial^2 u_x}{\partial x\partial z}+\frac{\partial^2 u_y}{\partial y\partial z}+\frac{\partial^2 u_z}{\partial z^2}\right)+G\left(\frac{\partial^2 u_z}{\partial x^2}+\frac{\partial^2 u_z}{\partial y^2}+\frac{\partial^2 u_z}{\partial z^2}\right)$$

$$\Rightarrow$$

1) $\rho\frac{\partial^2 u_x}{\partial t^2} = (\lambda+G)\frac{\partial \varepsilon_v}{\partial x}+G\nabla^2 u_x$

2) $\rho\frac{\partial^2 u_y}{\partial t^2} = (\lambda+G)\frac{\partial \varepsilon_v}{\partial y}+G\nabla^2 u_y$

3) $\rho\frac{\partial^2 u_z}{\partial t^2} = (\lambda+G)\frac{\partial \varepsilon_v}{\partial z}+G\nabla^2 u_z$

Differentiating 1), 2), 3) with respect to x, y, z respectively and adding:

$$\rho\frac{\partial^2 \varepsilon_{xx}}{\partial t^2} = (\lambda+G)\frac{\partial^2 \varepsilon_v}{\partial x^2}+G\nabla^2\varepsilon_{xx}$$

$$\rho\frac{\partial^2 \varepsilon_{yy}}{\partial t^2} = (\lambda+G)\frac{\partial^2 \varepsilon_v}{\partial y^2}+G\nabla^2\varepsilon_{yy}$$

$$\rho\frac{\partial^2 \varepsilon_{zz}}{\partial t^2} = (\lambda+G)\frac{\partial^2 \varepsilon_v}{\partial z^2}+G\nabla^2\varepsilon_{zz}$$

$$\xrightarrow{\text{adding}}\quad \rho\frac{\partial^2 \varepsilon_v}{\partial t^2} = (\lambda+G)\nabla^2\varepsilon_v+G\nabla^2\varepsilon_v \quad\Rightarrow\quad \rho\frac{\partial^2 \varepsilon_v}{\partial t^2} = (\lambda+2G)\nabla^2\varepsilon_v$$

Three-dimensional wave propagation

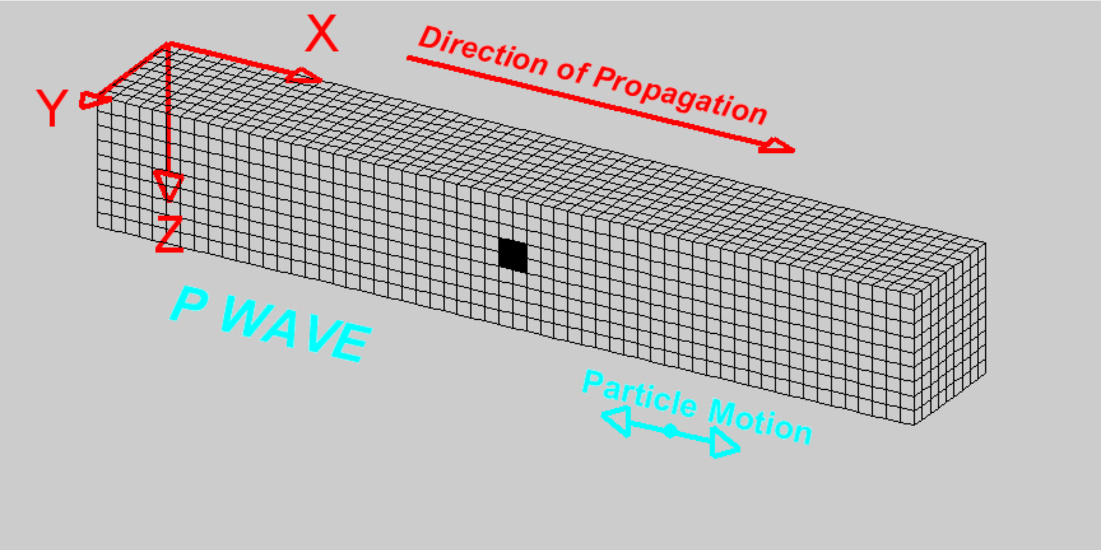
$$\rho \frac{\partial^2 \varepsilon_v}{\partial t^2} = (\lambda + 2G)\nabla^2 \varepsilon_v \quad \Rightarrow \quad \frac{\partial^2 \varepsilon_v}{\partial t^2} = \boxed{\frac{\lambda + 2G}{\rho}} \nabla^2 \varepsilon_v$$

During the propagation of a perturbation (wave), the second time derivative of the strain is proportional to its second spatial derivative, with a proportionality constant that has the dimensions of a squared velocity.

$$V_p^2 = \frac{(\lambda + 2G)}{\rho} \rightarrow V_p = \sqrt{\frac{\lambda + 2G}{\rho}}$$

Or in terms of K and G moduli:

$$\lambda = K - \frac{2}{3}G \rightarrow V_p = \sqrt{\frac{K + \frac{4}{3}G}{\rho}}$$



Source: IRIS (Incorporated Research Institutions for Seismology)

Three-dimensional wave propagation

Starting from the equation of motion and differentiating 2) and 3) with respect to z and y respectively:

1) $$\rho\frac{\partial^2 u_x}{\partial t^2} = (\lambda + G)\frac{\partial \varepsilon_v}{\partial x} + G\nabla^2 u_x$$

2) $$\rho\frac{\partial^2 u_y}{\partial t^2} = (\lambda + G)\frac{\partial \varepsilon_v}{\partial y} + G\nabla^2 u_y$$

3) $$\rho\frac{\partial^2 u_z}{\partial t^2} = (\lambda + G)\frac{\partial \varepsilon_v}{\partial z} + G\nabla^2 u_z$$

$$\Rightarrow$$

2a) $$\rho\frac{\partial^2}{\partial t^2}\frac{\partial u_y}{\partial z} = (\lambda + G)\frac{\partial \varepsilon_v}{\partial y \partial z} + G\nabla^2\frac{\partial u_y}{\partial z}$$

3a) $$\rho\frac{\partial^2}{\partial t^2}\frac{\partial u_z}{\partial y} = (\lambda + G)\frac{\partial \varepsilon_v}{\partial y \partial z} + G\nabla^2\frac{\partial u_z}{\partial y}$$

Subtracting eq. 3a and 2a:

$$\rho\frac{\partial^2}{\partial t^2}\underbrace{\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right)} = G\nabla^2\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right) \Rightarrow \rho\frac{\partial^2 \omega_x}{\partial t^2} = G\nabla^2 \omega_x$$

ω_x: rotation around x-axis

Three-dimensional wave propagation

Doing the same manipulation for each couple of motion equation:

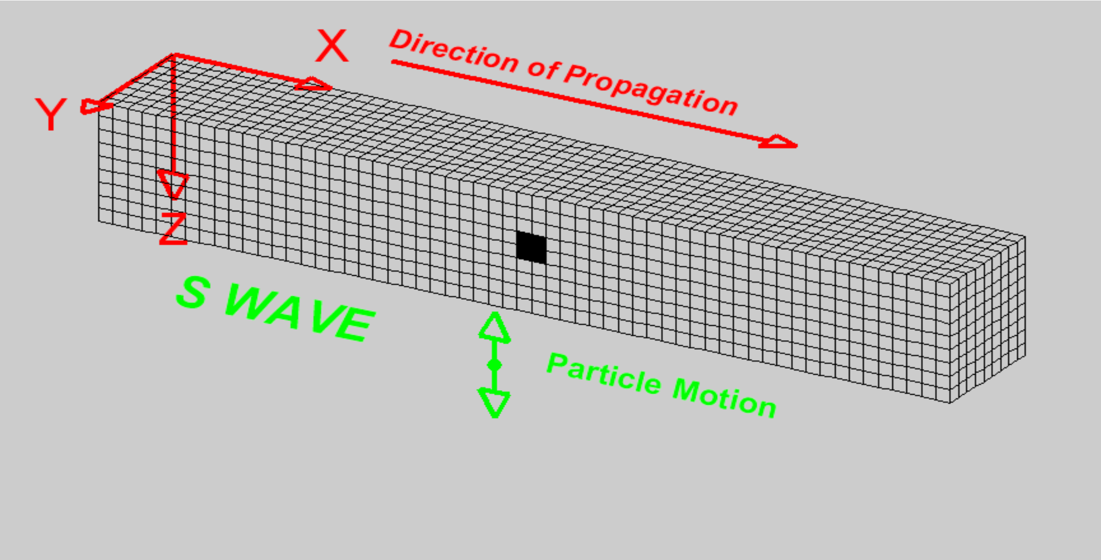
$$\rho \frac{\partial^2 \omega_x}{\partial t^2} = G\nabla^2 \omega_x \quad \Rightarrow \quad \frac{\partial^2 \omega_x}{\partial t^2} = \boxed{\frac{G}{\rho}} \nabla^2 \omega_x$$

$$\rho \frac{\partial^2 \omega_y}{\partial t^2} = G\nabla^2 \omega_y \quad \Rightarrow \quad \frac{\partial^2 \omega_x}{\partial t^2} = \boxed{\frac{G}{\rho}} \nabla^2 \omega_x$$

$$\rho \frac{\partial^2 \omega_z}{\partial t^2} = G\nabla^2 \omega_z \quad \Rightarrow \quad \frac{\partial^2 \omega_x}{\partial t^2} = \boxed{\frac{G}{\rho}} \nabla^2 \omega_x$$

During the propagation of a perturbation (wave), the second time derivative of the strain is proportional to its second spatial derivative, with a proportionality constant that has the dimensions of a squared velocity.

$$V_s^2 = \frac{G}{\rho} \rightarrow V_s = \sqrt{\frac{G}{\rho}}$$

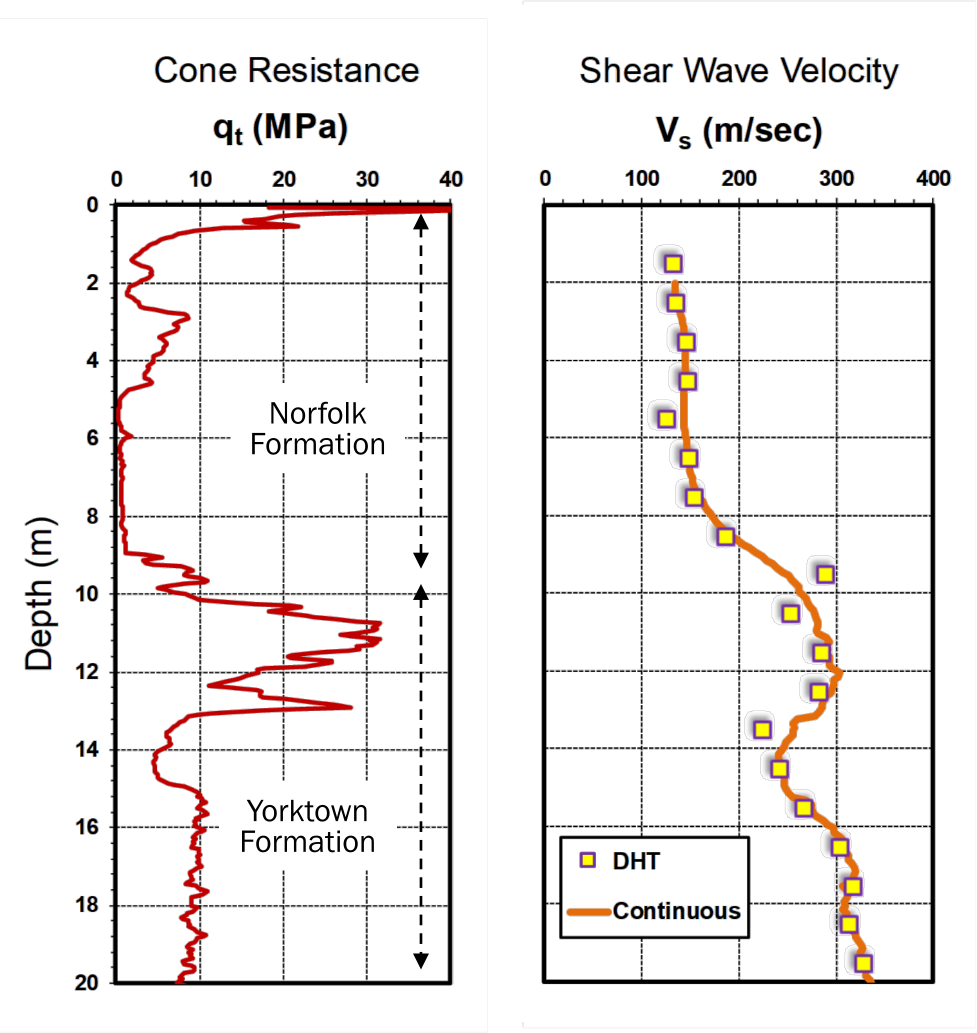


Source: IRIS (Incorporated Research Institutions for Seismology)

Small strain non-linear elasticity

- In case of satured soils, P-waves are related to the volumetric stiffness of water so it is difficult to obtain information on the solid skeleton.

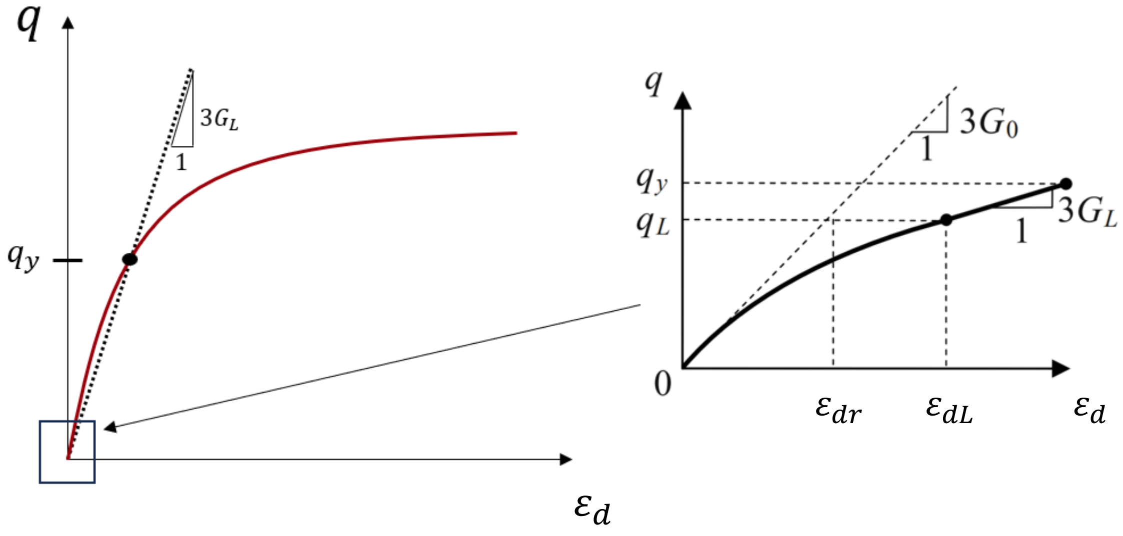
- On the contrary S-waves are used to obtain information on the stiffness of the solid skeleton since water doesn't have shear stiffness.

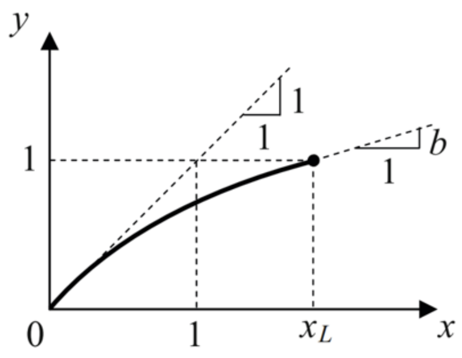


P. W. Mayne 2020

Small strain non-linear elasticity

- The range of small strains is identified in elastic domain
- Nonlinearity of shear modulus is modeled until a limit value of stress q_L; after this value is assumed a constant shear modulus until yielding (q_y).

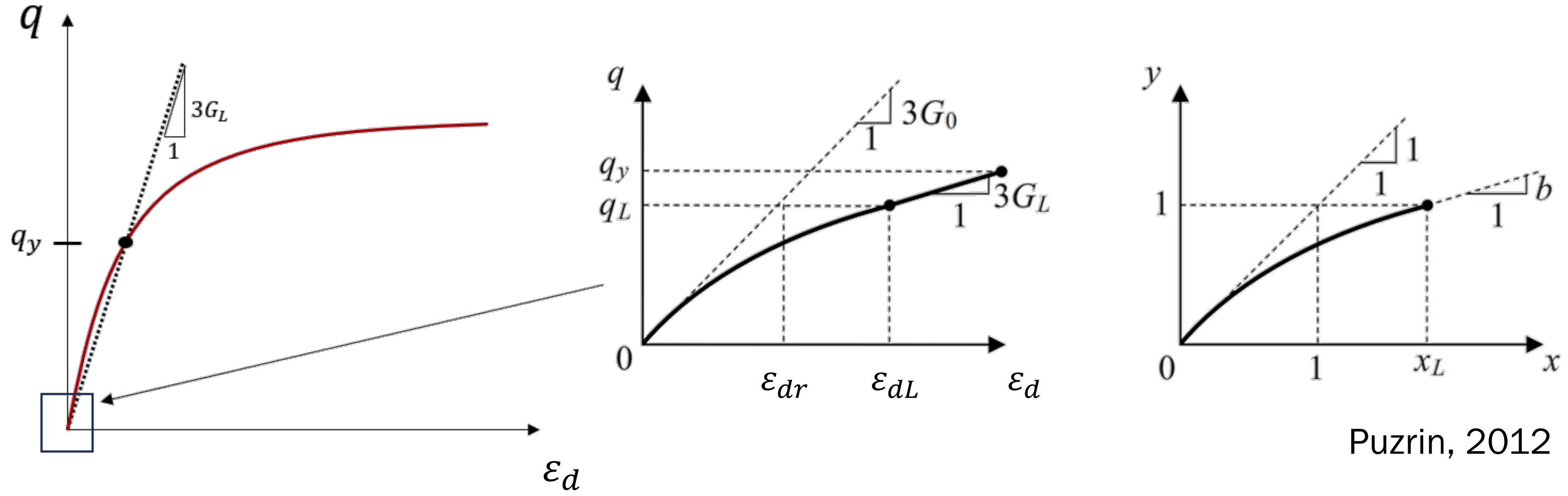




Puzrin, 2012

- G_0 initial tangent shear modulus
- ε_{dL} limit of deviatoric small strain
- G_L tangent shear modulus at q_L

Small strain non-linear elasticity



Puzrin, 2012

- G_0 initial tangent shear modulus
- ε_{qL} limit of deviatoric small strain
- G_L tangent shear modulus at q_L

- It is convenient use the following normalisation of the stresses and strains:

$$y = \frac{q}{q_L} \qquad x = \frac{\varepsilon_d}{\varepsilon_{dr}} \qquad \varepsilon_{dr} = \frac{q_L}{3G_0} \qquad x_L = \frac{\varepsilon_{dL}}{\varepsilon_{dr}}$$

$$\frac{dy}{dx} = \frac{1}{3G_0}\frac{dq}{d\varepsilon_d} = \frac{G_t(x)}{G_0}$$

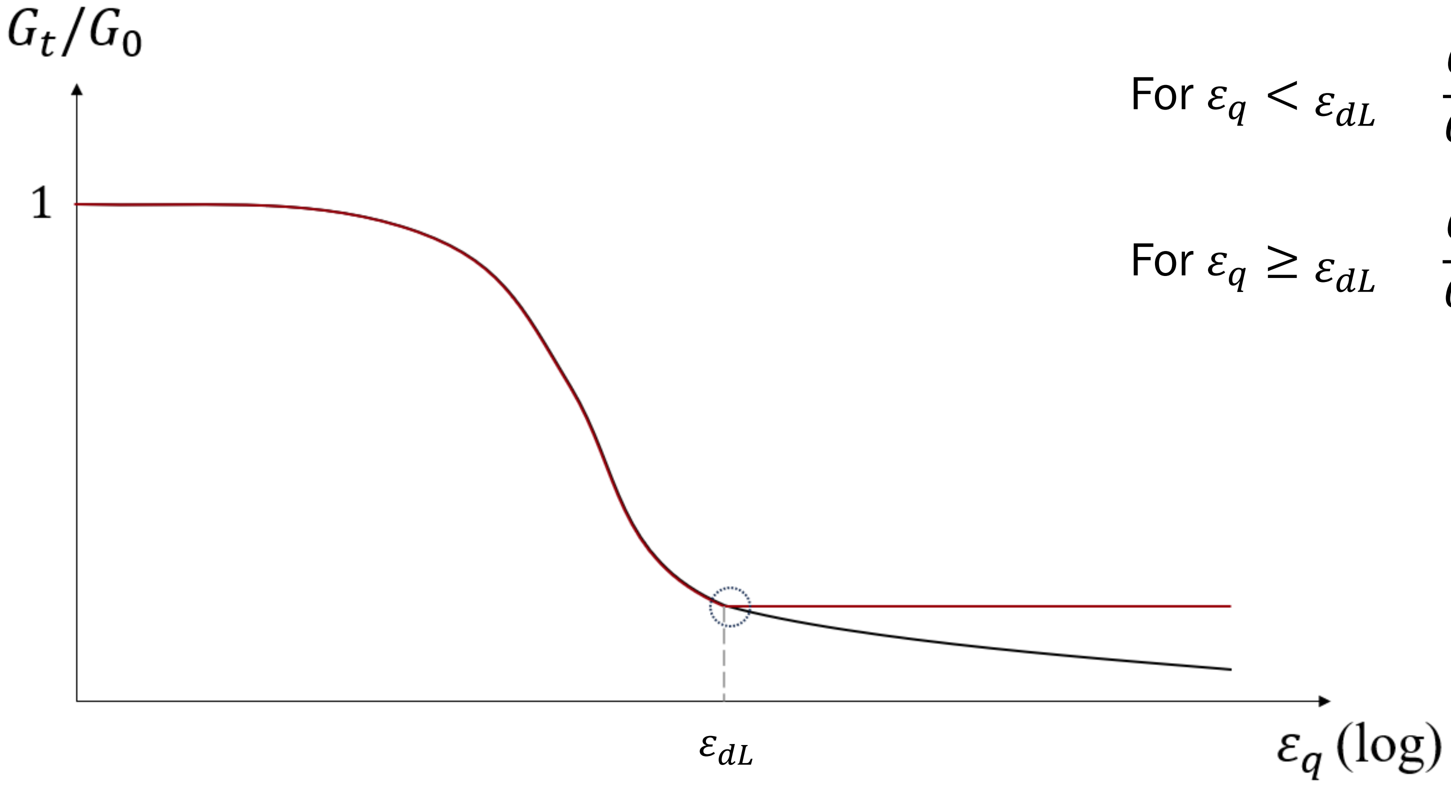
$G_t(x)$ tangent shear modulus

- The normalized analytical function has to satisfy :

in $x = 0$ and $y = 0$ $\quad \dfrac{dy}{dx} = 1$

in $x = x_L$ and $y = 1$ $\quad \dfrac{dy}{dx} = b = \dfrac{G_L}{G_0}$

Small strain non-linear elasticity



For $\varepsilon_q < \varepsilon_{dL}$ $\quad \dfrac{G_t}{G_0} = \dfrac{dy}{dx}$

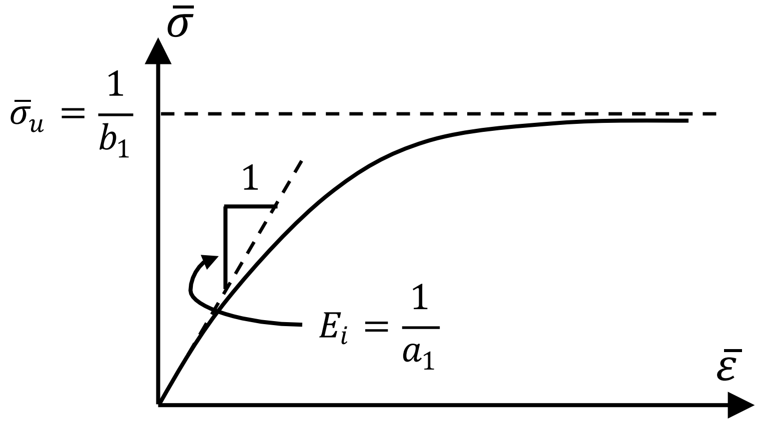
For $\varepsilon_q \geq \varepsilon_{dL}$ $\quad \dfrac{G_t}{G_0} =$ constant (until the yielding)

Small strain non-linear elasticity

There are some analytical functions for curve-fitting the deviatoric stress-strain behaviour of soils at small strains :

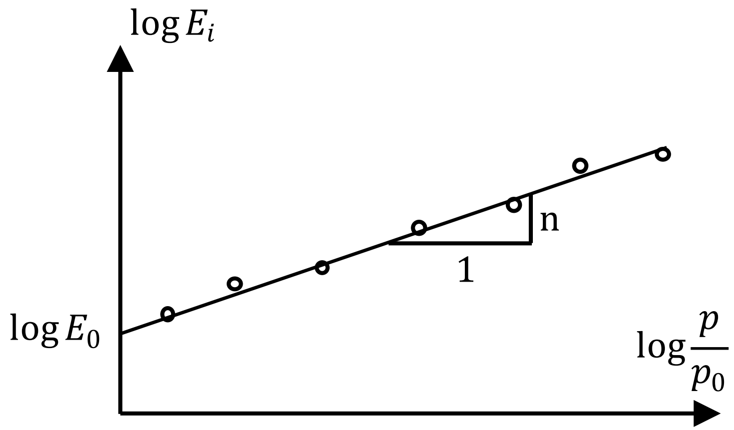
- Hyperbolic model (Kondner, 1963)

$$\bar{\sigma} = \frac{\bar{\varepsilon}}{a_1 + b_1 \bar{\varepsilon}} \qquad E_t = \frac{d\bar{\sigma}}{d\bar{\varepsilon}} = \frac{a_1}{(a_1 + b_1 \bar{\varepsilon})^2}$$



- Mean pressure dependency model (Duncan & Chang, 1970)

$$E_i = E_0 \cdot \left(\frac{p}{p_0}\right)^n \qquad \log E_i = n \log \frac{p}{p_0} + \log E_0$$



Summary

- In Geomechanics, geomaterials are viewed as engineering materials and their behaviour are explained by constitutive models.

- Basic constitutive model for geomaterials: Linear and non-linear elasticity.

- Even simple, elasticity can be still used in some engineering cases to appropriately address the behaviour of geomaterials.

Thank you for your attention